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ASYMPTOTIC PROBLEM OF BEAM STABILITY. LOSS OF STABILITY IN BENDING AND BUCKLING

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The present article is a continuation of [1], and is concerned with the stability of a thin (diameter $\varepsilon \to 0$) beam considered initially as a three-dimensional body. The limiting problem for a one-dimensional structure (beam) is obtained on the basis of an asymptotic expansion of the solution of a problem in elasticity theory in the limit to small diameters. As in [1], the elastic constants of the material are of order ε^{-4}, which ensures that the beam will have non-vanishing flexural rigidity as $\varepsilon \to 0$. In [1] the classical problem of the theory of beam stability was found under the condition that the initial stresses in the beam are of order ε^{-2} (i.e., the reciprocal of the length of the diameter of the beam). In this problem the loss of stability is caused by the presence of a non-vanishing axial compressive force [1]: $\langle \sigma_{11}{}^{*(-2)} \rangle \neq 0$ (the angular brackets denote the average computed with respect to an element of the structure of the beam; see below). What might be expected if the axial force is zero? In this case initial stresses that induce a loss of stability may arise in the beam (understood as a three-dimensional body). It seems reasonable to suppose that such self-balancing stresses would be so large in magnitude as to induce a loss of stability. In the present article this hypothesis is supported by considering initial stresses of order ε^{-3} with condition $\langle \sigma_{11}{}^{*(-3)} \rangle = 0$. Below it will be shown that in the situation we are considering, a loss of stability may by initiated by the moments of the initial stresses. The investigation of initial stresses of even higher orders represents a separate problem (the latter problem is discussed in [5-8], though not for thin bodies).

We analyze the problem on the basis of a two-scale method in a version for beams [1-4] that allows us to consider both nonhomogeneous beams of periodic structure as well as homogeneous, cylindrical beams [3] as a special case.

1. Statement of Problem. We wish to consider a region Ω_ε obtained by periodic repetition of a periodicity cell P_e, along the axis Ox_1 from $-a$ to a (see Fig. 1). The characteristic dimension of the periodicity cell $\varepsilon \ll 1$, a fact which is formally stated in the form $\varepsilon \to 0$. As $\varepsilon \to 0$, the region Ω_ε shrinks to the closed interval $[-a, a]$ on the axis Ox_1.

By the results of [9], the equilibrium problem for a body with initial stresses has the form

$$\int\limits_{\Omega_\varepsilon} \sigma^\varepsilon_{ij} \frac{\partial v_i}{\partial x_j} \, d\mathbf{x} = 0 \tag{1.1}$$

for any test function $\mathbf{v} \in V(\Omega_\varepsilon) = \{\mathbf{v} \in H^1(\Omega_\varepsilon): \mathbf{v}(\mathbf{x}) = 0 \text{ for } x_1 = \pm a\}$. (For the definition of these classes of functions, see [4].) The relationship between the current stresses $\sigma_{ij}{}^\varepsilon$, displacements $\mathbf{u}^\varepsilon$, and initial stresses $\sigma_{ij}{}^{*(-3)}$ is as follows:

$$\sigma^\varepsilon_{ij} = (\varepsilon^{-4} a_{ijkl}(\mathbf{x}/\varepsilon) + \varepsilon^{-3} b_{ijkl}(x_1, \mathbf{x}/\varepsilon)) \frac{\partial u^\varepsilon_k}{\partial x_l}. \tag{1.2}$$

Here $\varepsilon^{-4} a_{ijkl}(\mathbf{x}/\varepsilon)$ is the tensor of the elastic constants; $b_{ijkl}(x_1, \mathbf{x}/\varepsilon) = \sigma_{jl}{}^{*(-3)}(x_1, \mathbf{x}/\varepsilon)\delta_{ik}$; and δ_{ik} is the Kronecker symbol [9].

Accordingly, the following condition is imposed on the initial stresses:

$$\langle \sigma^{*(-3)}_{jl} \rangle = 0, \tag{1.3}$$

where $\langle \cdot \rangle = \dfrac{1}{m} \int\limits_{P_1} \cdot \, d\mathbf{y}$ is the average taken over the periodicity cell P_1 (see Fig. 1).

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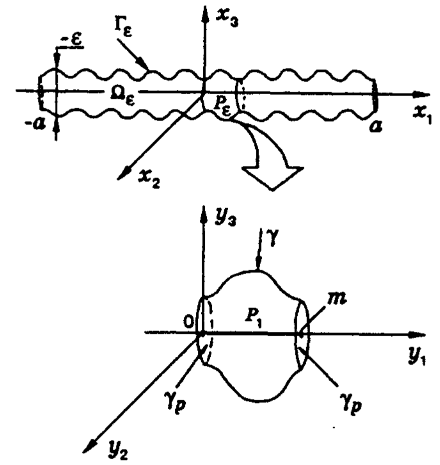

Fig. 1

2. Method of Asymptotic Expansion. In the present study we are considering a loss of stability with respect to global forms, i.e., associated with the appearance of global shifts u^ε accompanied by stable deformations within the periodicity cell. Asymptotic expansions of the displacements and stresses, and of the test function are selected in the form proposed in [2] for the case of beams:

$$\mathbf{u}^\varepsilon = \mathbf{u}^{(0)}(x_1) + \varepsilon \mathbf{u}^{(1)}(x_1, \mathbf{y}) + \ldots = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{u}^{(n)}(x_1, \mathbf{y}); \tag{2.1}$$

$$\sigma_{ij}^\varepsilon = \varepsilon^{-4} \sigma_{ij}^{(-4)}(x_1, \mathbf{y}) + \ldots = \sum_{m=-4}^{\infty} \varepsilon^m \sigma^{(m)}(x_1, \mathbf{y}); \tag{2.2}$$

$$\mathbf{v} = \mathbf{v}^{(0)}(x_1) + \varepsilon \mathbf{v}^{(1)}(x_1, \mathbf{y}) + \ldots = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{v}^{(n)}(x_1, \mathbf{y}) \tag{2.3}$$

(here $y = x/\varepsilon$ are rapid (local) variables, and $x_1 \in [-a, a]$ is a slow variable). The functions on the right sides of (2.1)-(2.3) are assumed to be periodic with respect to y_1 with period m (m is the length of the periodicity cell $P_1 = \{y = x/\varepsilon;\ x \in P_\varepsilon\}$ along the axis Oy_1; see Fig. 1).

The derivative of a function of the form $f(x_1, y)$ is calculated by substitution of the differentiation operation according to the following rule [2]:

$$\frac{\partial f}{\partial x_1} \to f_{,1x} + \varepsilon^{-1} f_{,1y}, \qquad \frac{\partial f}{\partial x_\alpha} \to \varepsilon^{-1} f_{,\alpha y} \qquad (\alpha = 2, 3).$$

Here and below the Roman subscripts assume values of 1, 2, and 3, while the Greek subscripts, values of 2 and 3.

Substituting (2.1) and (2.2) into (1.2) and recalling the differentiation rule just presented, we have

$$\sum_{m=-4}^{\infty} \varepsilon^m \sigma_{ij}^{(m)} = \sum_{n=0}^{\infty} \varepsilon^n (\varepsilon^{-4} a_{ijkl} + \varepsilon^{-3} b_{ijkl})(u_{k,1x}^{(n)} \delta_{1l} + \varepsilon^{-1} u_{k,ly}^{(n)}). \tag{2.4}$$

Here and below we are using the notation $1x = \partial/\partial x_1$ and $jy = \partial/\partial y_j$.

Equating terms with identical powers of ε, we obtain, in particular, the following relations:

in the case ε^{-4}:

$$\sigma_{ij}^{(-4)} = a_{ijk1}(\mathbf{y}) u_{k,1x}^{(0)} + a_{ijkl}(\mathbf{y}) u_{k,ly}^{(1)}; \tag{2.5}$$

in the case ε^{-3}:

$$\sigma_{ij}^{(-3)} = a_{ijk1}(\mathbf{y})u_{k,1x}^{(1)} + a_{ijkl}(\mathbf{y})u_{k,ly}^{(2)} + b_{ijk1}(x_1,\mathbf{y})u_{k,1x}^{(0)} + b_{ijkl}(x_1,\mathbf{y})u_{k,ly}^{(1)}; \tag{2.6}$$

in the case ε^{-2}:

$$\sigma_{ij}^{(-2)} = a_{ijk1}(\mathbf{y})u_{k,1x}^{(2)} + a_{ijkl}(\mathbf{y})u_{k,ly}^{(3)} + b_{ijk1}(x_1,\mathbf{y})u_{k,1x}^{(1)} + b_{ijkl}(x_1,\mathbf{y})u_{k,ly}^{(2)}. \tag{2.7}$$

Note that the quantities $\sigma_{ij}{}^{\varepsilon}$, $\sigma_{ij}{}^{(-4)}$, $\sigma_{ij}{}^{(-3)}$ and $\sigma_{ij}{}^{(-2)}$ determined by the formulas (1.2) and (2.5)-(2.7) are not symmetric with respect to ij. Moreover, in these formulas the following groups of terms may be identified: terms that are symmetric with respect to ij (convolution of a_{ijkl} with the corresponding expressions) and those which are not symmetric (convolution of b_{ijkl} with the corresponding expressions).

Substitution of (2.2) and (2.3) into the equilibrium equation (1.1) yields, for an appropriate choice of the test function, the following relation [2]:

$$\sigma_{ij,jy}^{(m)} = 0 \text{ in } \Omega_1^{\varepsilon}, \quad \sigma_{ij}^{(m)} n_j = 0 \text{ on } \Gamma_1^{\varepsilon}, \tag{2.8}$$

where $\Omega_1{}^{\varepsilon} = \{(x_1, y_2, y_3): \mathbf{x} \in \Omega_{\varepsilon}\}$; and $\mathbf{n}$ is the normal to the lateral surface $\Gamma_1{}^{\varepsilon}$ of the region $\Omega_1{}^{\varepsilon}$.

3. Equilibrium Equations. It was already noted in [10, 11] that, in using the asymptotic method, the equilibrium equations are obtained independently of the stress-deformation relation, in our case, independently of (1.2). In the case of beams, the equations for the axial forces $N_{ij}{}^{(m)} = \langle\sigma_{ij}{}^{(m)}\rangle$ and the moments $M_{i\alpha} = \langle\sigma_{i1}{}^{(m)}y\alpha\rangle$ are presented in [2]. Let us present the equilibrium equation from [2] which we will be using below:

$$N_{11,1x}^{(-3)} = 0, \quad N_{i1,1x}^{(-2)} = 0, \quad -M_{\alpha i,1x}^{(-3)} + N_{i\alpha}^{(-2)} = 0. \tag{3.1}$$

4. Relation Between Forces (and moments) and the Deformation Characteristics. The special features of the type of asymptotic problem which we are considering become clear when studying the governing relations. As will be clear in the subsequent presentation, in this case the existence of initial stresses has an effect on the equations which relate the forces and moments with the deformation characteristics of the beam as well as on the procedure used to eliminate the unknown functions (which, in the case of classical beams, have the meaning of cross forces) from the equilibrium equations. There is no such state in the case of unstressed beams [2].

Let us consider problem (2.8) with m = −4. By (2.5),

$$\sigma_{ij}^{(-4)} = a_{ijkl}(\mathbf{y})u_{k,ly}^{(1)} + a_{ijk1}(\mathbf{y})u_{k,1x}^{(0)}(x_1). \tag{4.1}$$

The solution of this problem is presented in [2]:

$$\mathbf{u}^{(1)} = -y_\alpha u_{\alpha,1x}^{(0)}(x_1)\,\mathbf{e}_1 + y_{\tilde\beta}\, s_\beta\, \mathbf{e}_\beta\, \varphi(x_1) + \mathbf{V}(x_1). \tag{4.2}$$

Here $\varphi(x_1)$ is an arbitrary function (having the sense of beam buckling); $\mathbf{V}(x_1)$, an arbitrary function (having the sense of an axial displacement); $s_1 = 0$; $s_2 = -1$; and $s_3 = 1$; and $\{\mathbf{e}_i\}$ and the basis vectors of a standard rectangular coordinate system;

$$.\alpha,\beta = 2,3; \; \tilde\beta = \begin{cases} 2 & \text{if} \quad \beta = 3, \\ 3 & \text{: if} \quad \beta = 2. \end{cases}$$

Substitution of (4.2) into (2.6) yields

$$
\sigma_{ij}^{(-3)} = -a_{ij11}(\mathbf{y})y_\alpha u^{(0)}_{\alpha,1x1x}(x_1) + a_{ijk1}(\mathbf{y})V_{k,1x}(x_1) + a_{ijkl}(\mathbf{y})u^{(2)}_{k,ly} +
$$

$$
+a_{ij\beta 1}(\mathbf{y})s_\beta\, y_{\bar\beta}\, \varphi_{,1x}(x_1) + b_{ij\alpha_1}(x_1,\mathbf{y})u^{(0)}_{\alpha,1x}(x_1) -
$$

$$
-b_{ij1\alpha}(x_1,\mathbf{y})u^{(0)}_{\alpha,1x}(x_1) + b_{ij\beta\bar\beta}(x_1,\mathbf{y})\, s_\beta \varphi(x_1). \tag{4.3}
$$

Now let us consider problem (2.8) with m = -3, where $\sigma_{ij}{}^{(-3)}$ are determined by (4.3).

Proposition 1. Suppose that the initial stresses satisfy the equilibrium equations

$$
\frac{\partial \sigma^*_{ij}}{\partial x_j} = 0 \text{ in } \Omega_\varepsilon, \quad \sigma^*_{ij} n_j = 0 \text{ on } \Gamma_\varepsilon; \tag{4.4}
$$

whence

$$
b_{ij\alpha 1,jy} = b_{ij1\alpha,jy} = b_{ij\beta\bar\beta,jy} = 0. \tag{4.5}
$$

The latter equality follows from the definition of b_{ijkl} and the following equalities:

$$
\sigma^{*(-3)}_{ij,jy} = 0 \text{ in } \Omega^\varepsilon_1, \quad \sigma^{*(-3)}_{ij} n_j = 0 \text{ on } \Gamma^\varepsilon_1, \tag{4.6}
$$

which are obtained from the equations presented earlier if expansions of the form $\sigma_{ij}{}^* = \varepsilon^{-3}\sigma_{ij}{}^{*(-3)} + \ldots$ are substituted into these equations and Remark 1 (see below) is noted.

By Proposition 1 the final terms in (4.3) (i.e., those which contain b_{ijkl}) may be omitted in solving problem (2.8); we will, in fact, do just that.

Remark 1. In considering problems involving variables y, the functions of the variable x_1 play the role of a parameter. The same thing occurs in the case of integration with respect to the variable y.

In view of this remark, the solution of problem (2.8) with m = -3, i.e., the result stated in (4.3) may be represented in the following form:

$$
\mathbf{u}^{(2)} = -\mathbf{N}^{2\alpha}(\mathbf{y})\, u^{(0)}_{\alpha,1x1x}(x_1) - y_\alpha V_{\alpha,1x}(x_1)\,\mathbf{e}_1 + \mathbf{N}^{11}(\mathbf{y})V_{1,1x}(x_1) + \mathbf{X}^3(\mathbf{y})\,\varphi(x_1), \tag{4.7}
$$

where $\mathbf{N}^{11}$, $\mathbf{N}^{2\alpha}$, and $\mathbf{X}^3$ are the solutions of the cell-based problems corresponding to tension, bending, and buckling, respectively, of a structural element of the bar; these solutions were introduced in [2]. Their explicit form in the problem which we are considering here is not essential, and, therefore, we will not describe it in detail in the present study. A detailed description of these functions is given in [2].

Following simple algebra, substitution of (4.7) into (4.3) yields the following:

$$
\sigma_{ij}^{(-3)} = (a_{ij11}(\mathbf{y}) + a_{ijkl}(\mathbf{y})\mathbf{N}^{11}_{k,ly}(\mathbf{y}))V_{1,1x}(x_1) - (a_{ij11}(\mathbf{y})y_\alpha +
$$

$$
+a_{ijkl}(\mathbf{y})\mathbf{N}^{2\alpha}_{k,ly}(\mathbf{y}))u^{(0)}_{\alpha,1x1x}(x_1) + (a_{ij\beta\bar\beta}(\mathbf{y})s_\beta y_\beta + a_{ijkl}(\mathbf{y})X^3_{k,ly}(\mathbf{y}))\varphi_{,1x}(x_1) + \tag{4.8}
$$

$$
+(b_{ij\alpha 1}(x_1,\mathbf{y}) - b_{ij1\alpha}(x_1,\mathbf{y}))u^{(0)}_{\alpha,1x}(x_1) + b_{ij\beta\bar\beta}(x_1,\mathbf{y})\, s_\beta\, \varphi(x_1).
$$

Let us integrate the latter expression with respect to the periodicity cell P_1, then multiply (4.8) by y_β, and finally integrate the result with respect to the periodicity cell P_1. Recalling that in integration with respect to y, the functions of the argument x_1 play the role of parameters, we find that

$$
N_{11}^{(-3)} = A^0_1 V_{1,1x} - A^1_{1\alpha} u^{(0)}_{\alpha,1x1x} + B^0_{11}\varphi_{,1x}; \tag{4.9}
$$

$$
M_{i\beta}^{(-3)} = {}^1A_{i\beta} V_{1,1x} - A^2_{i\beta\alpha} u^{(0)}_{\alpha,1x1x} + B^1_{i\beta}\varphi_{,1x} + B_{i\beta\alpha} u^{(0)}_{\alpha,1x} + J_{i\beta}\varphi. \tag{4.10}
$$

Proposition 2. If $\sigma_{ij}^{*(-3)}$ satisfy (4.6) and are periodic with respect to y_1 with period m, then $\langle\sigma_{i\alpha}^{*(-3)}\rangle = 0$, $\alpha = 2.3$.

To verify the assertion, we multiply the first equality in (4.6) by y_α and integrate the result by parts over the periodicity cell P_1, recalling the second equality in (4.6) and the fact that $\sigma_{ij}^{*(-3)}$ is periodic with respect to y_1 (the function y_α is also periodic with respect to y_1). As a result we obtain the desired result.

By Proposition 2 and Conditions (1.3) and (4.9), there are no terms present containing $u_{\alpha,1x}^{(0)}$ and φ.

The following notation was introduced in (4.9) and (4.10):

$$
\begin{aligned}
A_1^0 &= \langle a_{1111}(\mathbf{y}) + a_{11kl}(\mathbf{y})N_{k,ly}^{11}(\mathbf{y})\rangle,\\
A_{1\alpha}^1 &= \langle a_{1111}(\mathbf{y})y_\alpha + a_{11kl}(\mathbf{y})N_{k,ly}^{2\alpha}(\mathbf{y})\rangle,\\
{}^1A_{i\beta} &= \langle (a_{i111}(\mathbf{y}) + a_{i1kl}(\mathbf{y})N_{k,ly}^{11}(\mathbf{y}))\mathbf{y}_\beta\rangle,\\
A_{i\alpha\beta}^2 &= \langle (a_{i111}(\mathbf{y})y_\alpha + a_{i1kl}(\mathbf{y})N_{k,ly}^{2\alpha}(\mathbf{y}))y_\beta\rangle,\\
B_{ij}^0 &= \langle a_{ij\gamma\bar{\gamma}}(\mathbf{y})s_\gamma y_\gamma + a_{ijkl}(\mathbf{y})X_{k,ly}^3(\mathbf{y})\rangle,\\
B_{i\beta}^1 &= \langle (a_{i1\gamma\bar{\gamma}}(\mathbf{y})s_\gamma y_\gamma + a_{i1kl}(\mathbf{y})X_{k,ly}^3(\mathbf{y}))y_\beta\rangle,\\
B_{i\beta\alpha} &= \langle (b_{i1\alpha1} - b_{i11\alpha})y_\beta\rangle = \langle\sigma_{11}^{*(-3)}y_\beta\rangle\delta_{i\alpha} - \langle\sigma_{1\alpha}^{*(-3)}y_\beta\rangle\delta_{i1},\\
J_{i\beta} &= \langle b_{i1\gamma\bar{\gamma}}s_\gamma y_\beta\rangle = \langle\sigma_{1\bar{\gamma}}^{*(-3)}y_\beta\rangle s_\gamma\delta_{i\gamma}.
\end{aligned}
\tag{4.11}
$$

The first six equalities in (4.11) determine the degree of longitudinal, flexural, and torsional rigidity of the beam. Comparing these results with those of [2], it becomes clear that they are the same as for a beam without initial stresses. The last two equalities in (4.11) show that in the case under consideration the governing relations depend on the initial stresses, though the type of the dependence is not the same as that presented in [9]. The dependence presented in [9] arises when initial stresses of order ε^{-4} are considered.

The equilibrium equations (3.1) may be subdivided into the following groups:

$$N_{11,1x}^{(-3)} = 0; \tag{4.12}$$

$$N_{i1,1x}^{(-2)} = 0; \tag{4.13}$$

$$-M_{\alpha i,1x}^{(-3)} + N_{i\alpha}^{(-2)} = 0. \tag{4.14}$$

In (4.14) there is an undefined function $N_{ij}^{(-2)}$ present. The procedure of eliminating this function from (4.14) in the case when there are no pre-stresses present is as follows [2]: Eq. (4.14) is differentiated with $i = 1$

$$-M_{\alpha1,1x1x}^{(-3)} + N_{1\alpha,1x}^{(-2)} = 0, \tag{4.15}$$

and then (4.13) is applied with $i = \alpha$

$$N_{\alpha1,1x}^{(-2)} = 0.$$

If the quantities $N_{ij}^{(-2)}$ were symmetric with respect to ij (as is the case with the stresses $\sigma_{ij}^{*(-3)}$ or the forces when there are no initial stresses [2]), we would arrive at the classical equilibrium equation $-M_{\alpha1,1x1x}^{(-3)} = 0$. In the case we are considering there is no such symmetry. To eliminate $N_{1\alpha}^{(-2)}$ we isolate the nonsymmetric part $N_{1\alpha}^{(-2)}$ by representing $N_{1\alpha}^{(-2)}$ in the form $N_{1\alpha}^{(-2)} = N_{\alpha1}^{(-2)} + K_\alpha$ and note that, because the elastic constants a_{ijkl} are symmetric [12], there is a term in (2.7) containing the unknown function $u^{(3)}$, which is symmetric with respect to ij. Consequently, it is not necessary to know $u^{(3)}$ in order to be able to compute K_α.

As a result we obtain

$$K_\alpha = N^{(-2)}_{1\alpha} - N^{(-2)}_{\alpha 1} = \langle \sigma^{(-2)}_{1\alpha} - \sigma^{(-2)}_{\alpha 1} \rangle = \langle (b_{1\alpha k1} - b_{\alpha 1 k1}) u^{(1)}_{k,1x} \rangle + \langle (b_{1\alpha kl} - b_{\alpha 1 kl}) u^{(2)}_{k,ly} \rangle. \qquad (4.16)$$

Proposition 3. The stresses $\sigma_{ij}{}^{*(-3)}$, described in Propositions 1 and 2 satisfy the following equality:

$$\langle (b_{1\alpha kl} - b_{\alpha 1 kl})\, u^{(2)}_{k,ly} \rangle = 0. \qquad (4.17)$$

To prove the assertion, let us consider the quantity $\langle \sigma^{*(-3)}_{\gamma l} u^{(2)}_{k,ly} \rangle$. Integration by parts on P_1 shows that it has the form

$$-\int_{P_1} \sigma^{*(-3)}_{\gamma l,ly} u^{(2)}_k dy + \int_{\gamma} \sigma^{*(-3)}_{\gamma l} n_l u^{(2)}_k dy + \int_{\gamma_p} \sigma^{*(-3)}_{\gamma l} n_l u^{(2)}_k dy = 0.$$

The latter equality follows from (4.6) in view of the fact that the functions $\sigma_{\gamma l}{}^{*(-3)}$ and $u^{(2)}$ are periodic and the fact that the normals to the faces γ_p of the periodicity cell P_1 perpendicular to the axis Oy_1 (cf. Fig. 1) are oppositely directed. In view of the fact that $b_{ijkl} = \sigma_{jl}{}^{*(-3)}\delta_{ik}$, Eq. (4.17) reduces to the equality considered earlier.

Substituting the expression for $u^{(1)}$ from (4.2) and the expression for $u^{(2)}$ from (4.7) into (4.16) yields, in view of Proposition 1, the following result:

$$K_\alpha = \langle (b_{1\alpha k1} - b_{\alpha 1 k1}) u^{(1)}_{k,1x} \rangle = \langle (b_{1\alpha 11} - b_{\alpha 111}) y_\beta \rangle u^{(0)}_{\beta,1x}(x_1) +$$
$$+ \langle (b_{1\alpha\beta 1} - b_{\alpha 1\beta 1}) y_{\bar\beta} \rangle s_\beta \varphi_{,1x}(x_1) + \langle b_{1\alpha k1} - b_{\alpha 1 k1} \rangle V_{k,1x}(x_1).$$

Substituting $b_{ijkl} = \sigma_{jl}{}^{*(-3)}\delta_{ik}$ into this equality and using Proposition 2, we find that

$$K_\alpha = \langle \sigma^{*(-3)}_{\alpha 1} y_\beta \rangle u^{(0)}_{\beta,1x} - \langle \sigma^{*(-3)}_{11} y_{\bar\beta} \rangle \delta_{\alpha\beta} s_\beta \varphi_{,1x}. \qquad (4.18)$$

Let us now consider the equations for the moment of torsion $M = M_{32}{}^{(-3)} - M_{23}{}^{(-3)}$. The equilibrium equations for M are obtained directly from (4.14):

$$-M_{,1x} + K = 0.$$

Here

$$K = N^{(-2)}_{32} - N^{(-2)}_{23}.$$

Proceeding as before and recalling (2.7) and (4.2) and Proposition 3, we have

$$K = -\langle (b_{3211} - b_{2311}) y_\alpha \rangle u^{(0)}_{\alpha,1x1x}(x_1) + \langle (b_{32\beta 1} - b_{23\beta 1}) y_{\bar\beta} \rangle s_\beta \varphi_{,1x}(x_1) + \langle b_{32k1} - b_{23k1} \rangle V_{k,1x}(x_1).$$

In view of the fact that $b_{ijkl} = \sigma_{jl}{}^{*(-3)}\delta_{ik}$, and using Proposition 2, we find that

$$K = \langle \sigma^{*(-3)}_{21} y_2 + \sigma^{*(-3)}_{31} y_3 \rangle \varphi_{,1x}. \qquad (4.19)$$

Note that (4.18) and (4.19) may be represented in the form

$$K_\alpha = M^*_{\alpha\beta} u^{(0)}_{\beta,1x} + M^*_{1\bar\alpha} s_\alpha \varphi_{,1x}, \quad K = (M^*_{22} + M^*_{33}) \varphi_{,1x}, \qquad (4.20)$$

where $M_{\alpha\beta}{}^* = \langle \sigma_{\alpha 1}{}^{*(-3)} y_\beta \rangle$ are the prestressing moments.

The system of equations which results has the form

$$N^0(V_{1,1x}, u^{(0)}_{\beta,1x1x}, \varphi_{,1x})_{,1x} = 0; \qquad (4.21)$$

$$(M^0_\alpha(V_{1,1x}, u^{(0)}_{\beta,1x1x}, \varphi_{,1x}) + \lambda B_{1\alpha\beta} u^{(0)}_{\beta,1x})_{,1x1x} = -\lambda K_\alpha(u^{(0)}_{\beta,1x1x}, \varphi_{,1x})_{,1x}; \qquad (4.22)$$

$$(M^0(V_{1,1x}, u^{(0)}_{\beta,1x1x}, \varphi_{,1x}) + \lambda(B_{23\beta} - B_{32\beta}) u^{(0)}_{\beta,1x} + \lambda(J_{23} - J_{32})\varphi)_{,1x} = -\lambda K(\varphi_{,1x}). \qquad (4.23)$$

Here N^0, $M_\alpha{}^0$, and M^0 denote expressions that correspond to the problem without initial stresses (obtained from (4.9)-(4.11) with $\sigma_{ij}{}^{*(-3)} = 0$), while the remaining quantities are given by (4.18) and (4.19).

For the sake of ease of visualization, Eqs. (4.21)-(4.23) have been written for initial stresses that are proportional to the parameter λ, i.e., $\varepsilon^{-3} \lambda\sigma_{ij}{}^{*(-3)}$ (in the form of an eigenvalue problem).

For the coefficients on the left side of (4.22) and (4.23) we find, by (4.11),

$$\begin{aligned} B_{1\alpha\beta} &= -\langle \sigma^{*(-3)}_{\alpha 1} y_\beta \rangle = -M^*_{\alpha\beta}, \\ B_{23\beta} - B_{32\beta} &= \langle \sigma^{*(-3)}_{11} y_3 \rangle \delta_{2\beta} - \langle \sigma^{*(-3)}_{11} y_2 \rangle \delta_{3\beta} = M^*_{13}\delta_{2\beta} - M^*_{12}\delta_{3\beta}, \\ J_{23} - J_{32} &= -\langle \sigma^{*(-3)}_{31} y_3 \rangle - \langle \sigma^{*(-3)}_{21} y_2 \rangle = -M^*_{33} - M^*_{22}. \end{aligned} \qquad (4.24)$$

In the case where the rod ends are rigidly fastened, the boundary conditions for Eqs. (4.21)-(4.23) are as follows [2]:

$$V_1(\pm a) = u^{(0)}_\alpha(\pm a) = u^{(0)}_{\alpha,1x}(\pm a) = \varphi(\pm a) = 0. \qquad (4.25)$$

Thus, a complete boundary-value problem has been obtained for describing a beam as a one-dimensional structure.

Note that the coefficients of the operator that describes a loss of stability in (4.24) have the sense of the moments of the initial stresses.

Proposition 4. Suppose that the initial stresses satisfy the conditions of Proposition 1, whence

$$\langle \sigma^{*(-3)}_{i\alpha} y_\alpha \rangle = 0. \qquad (4.26)$$

To prove Eq. (4.26), we need only multiply the first equation in (4.6) by $y_\alpha{}^2$ and integrate the result by parts, recalling the boundary condition in (4.6) and the fact that $\sigma_{i\alpha}{}^{*(-3)}$, y_α and y_1 are periodic with respect to y_1.

Recalling that $\sigma_{ij}{}^{*(-3)}$ is symmetric with respect to ij, we have as a corollary of Proposition 4 in the case i = 1 the fact that $M_{\alpha\alpha}{}^* = \langle\sigma_{\alpha 1}{}^{*(-3)} y_\alpha\rangle = 0$. From the latter result, we find that K from (4.20) and $J_{23} - J_{32}$ from (4.24) both vanish. On the basis of the results that have been obtained and the results in (4.23) and (4.25), we conclude that in this case a purely torsional form of the loss of stability is not possible.

5. On the Calculation of the Initial Stresses. If we consider (4.11), we are led to remark that the coefficients $A_1{}^0$, $A_{1\alpha}{}^1$, ${}^1A_{i\beta}$, $A_{i\alpha\beta}{}^2$, $B_{ij}{}^0$, and $B_{i\beta}{}^1$ are calculated through solving the cell-based problems N^{11}, $N^{2\alpha}$, and X^3 which are directly related to the structure of the beam [1, 2, 4]. At the same time, the coefficients corresponding to a loss of stability do not contain these functions explicitly (see (4.20) and (4.24)). This does not mean that the coefficients of (4.20) and (4.24) are, in general, independent of the structure of the beam, only that the dependence of the coefficients on the beam structure is in the form of an implicit relation.

By the results of [2], the stresses $\sigma_{ij}{}^{*(-3)}$ may be expressed in terms of the solution of the cell-based problem and the solution of the boundary-value problem corresponding to (4.6). Substituting this expression into (4.20) and (4.24) yields explicit expressions for the coefficients of (4.20) and (4.24) in terms of the solution of the cell-based problem, In the case we are considering, it is best to determine only the moments $M_{\alpha\beta}{}^*$ from the solution of the boundary-value problem. Moreover, the dependence of $M_{\alpha\beta}{}^*$ on the structure of the beam becomes clear from the dependences of these moments on the coefficients $A_1{}^0$, $A_{1\alpha}{}^1$, ${}^1A_{i\beta}$, $A_{i\alpha\beta}{}^2$, $B_{ij}{}^0$, and $B_{ij}{}^1$.

For beams that are formed from such structure elements as beams, rods, etc., a circumstance that is also encountered often, the methods that are proposed in [13, 14] may be used to solve the cell-based problems.

In connection with the results that have been obtained here, note that questions related to the loss of stability of nonsymmetric (compound) beams have been considered in structural mechanics (see, for example, [15]) on the basis of the method of hypotheses. There is a degree of analogy between the formulas presented in [15] and those of (4.21)-(4.23), though this analogy does not extend beyond the fact that there are expressions containing derivatives of different orders in the terms occurring in the equations that correspond to a loss of stability. One difference from the results obtained in [15] worth noting is the fact that the formulas (4.21)-(4.23) are valid for homogeneous (non-compound) beams that are experiencing bending and twisting moments.

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